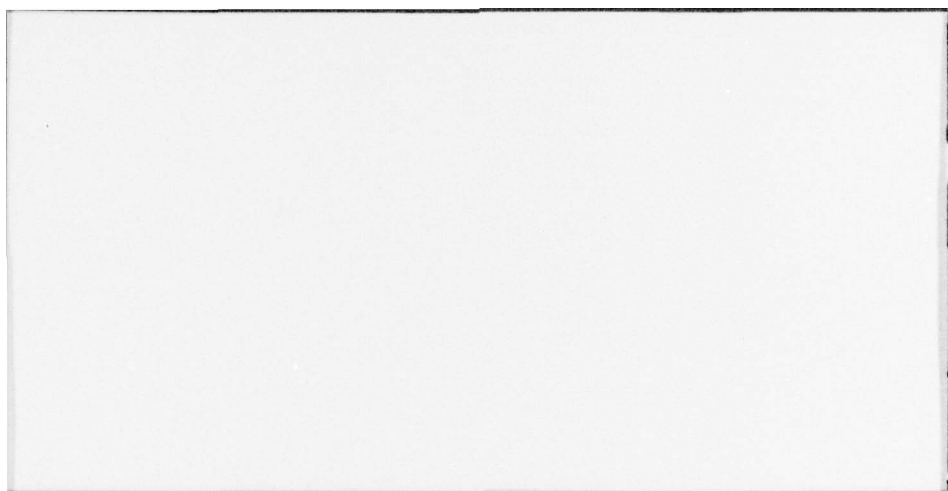END

DATE
FILMED
7-77

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Steady-State Wave Propagation in Simple and Compound

ACOUSTIC WAVEGUIDES

J. C. Guillot and C. H. Wilcox

Technical Summary Report #31

April 1977

DISTRIBUTION STATEMENT A

Approved for public release;
Distribution Unlimited

Prepared under Contract No. N00014-76-C-0276

Task No. NR-041-370

for

Office of Naval Research

D D C

JUN 28 1977

A

ABSTRACT

A limiting absorption principle and corresponding existence theorem is presented for steady-state acoustic wave fields in simple and compound waveguides. The simple guides are semi-infinite cylinders, sectoral horns and conical horns. The compound guides are structures built by coupling one or more simple guides to a bounded cavity. The theory is developed under minimal regularity hypotheses about the waveguide walls. The class of allowable guides includes all the simple, but non-smooth, structures that arise in applications such as guides built from polyhedra and sections of cylinders, cones, spheres and other simple geometrical structures.

ii

CONTENTS

SECTION ONE

INTRODUCTION

The acoustic waveguides studied in this paper are characterized geomet-
rically by a class of unbounded domains Ω in 3-dimensional Euclidean space
E^3 of the form

$$(1.1) \qquad \Omega = \Omega_0 \cup \left(\bigcup_{i=1}^{I} S_i \right) \cup \left(\bigcup_{j=1}^{I'} S_j' \right) \cup \left(\bigcup_{k=1}^{I''} S_k'' \right)$$

where Ω_0 is bounded, each S_i is a semi-infinite cylinder, each S_j' is a
sectoral horn and each S_k'' is a conical horn. Each set S_i, S_j', S_k'' is
defined by a coordinate system in E^3. In particular

$$(1.2) \qquad S_i = S_i(y_0^i) = \{q \in E^3: \ (x_1^i(q), x_2^i(q)) \in G_i \text{ and } y^i(q) > y_0^i\}$$

where $(x_1^i, x_2^i, y^i) \in R^3$ are Cartesian coordinates, $y_0^i \in R$ and $G_i \subset R^2$ is a
bounded domain. Similarly,

$$(1.3) \qquad S_j' = S_j'(\rho_0^j) = \{q \in E^3: \ \rho^j(q) > \rho_0^j \text{ and } (\theta^j(q), z^j(q)) \in G_j'\}$$

where $(\rho^j, \theta^j, z^j) \in R^3$ are cylindrical coordinates, $\rho_0^j \geq 0$, $G_j' = M_j' \times (0, z_0^j)$,
M_j' is a domain in the unit circle S^1 (which may coincide with S^1) and $z_0^j > 0$.
Finally,

$$(1.4) \qquad S_k'' = S_k''(r_0^k) = \{q \in E^3: \ r^k(q) > r_0^k \text{ and } \eta^k(q) \in G_k''\}$$

where $(r^k, \eta^k) \in R \times S^2$ are spherical coordinates, S^2 is the unit 2-sphere,
$r_0^k \geq 0$ and G_k'' is a domain in S^2. The pair-wise intersections of the

2

I + I' + I" sets S_i, S_j', S_k'' are assumed to be bounded. For suitable choices of Ω_0, y_0^i, ρ_0^j, r_0^k one then has

(1.5)   the I + I' + I" sets S_i, S_j', S_k'' are pair-wise disjoint

By further specialization of Ω_0, y_0^i, ρ_0^j, r_0^k it can be arranged that

(1.6)
the I + I' + I" + 1 sets Ω_0, $S_i(y_0^i + 1)$, $S_j'(\rho_0^j + 1)$, $S_k''(r_0^k + 1)$ are pair-wise disjoint.

Throughout the paper Ω is assumed to satisfy (i.1)-(1.6). The simple wave-guides are the special cases where I + I' + I" = 1 and Ω_0 is empty. The remaining cases are called compound waveguides. The domains Ω include a large number of geometries that arise in applications. The sectoral horns include parallel plate waveguides ($M_j' = S^1$). The conical horns include the exteriors of spheres ($G_k'' = S^2$), half-spaces, wedges and the complement of a plane with a bounded aperture.

Each waveguide is assumed to be filled with a homogeneous fluid whose equilibrium state is characterized by a sound speed c_0, density ρ_0 and pressure p_0. It is assumed here that $c_0 = 1$ and $\rho_0 = 1$ since this can be achieved by a suitable choice of units. The acoustic field in Ω is char-acterized by a real-valued acoustic potential u(t,q) which satisfies the d'Alembert equation

(1.7) $$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(t,q), \; t \in R, \; q \in \Omega$$

where Δ is the Laplacian in E^3. The function f will be called the source function. It has the structure [19, p. 280]

$$(1.8) \qquad\qquad f(t,q) = \partial V(t,q)/\partial t + Q(t,q)$$

where $V(t,q)$ is a potential for the (conservative) force field and $Q(t,q)$ is the volume flow in Ω. On the waveguide walls, described by the boundary $\partial\Omega$, $u(t,q)$ satisfies the Dirichlet condition

$$(1.9) \qquad\qquad u\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a soft boundary}$$

and the Neumann condition

$$(1.10) \qquad\qquad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \text{ if } \partial\Omega \text{ is a hard boundary}$$

where $\partial u/\partial \nu = \nabla u \cdot \vec{\nu}$ and $\vec{\nu}$ is a normal vector on $\partial\Omega$.

Steady-state acoustic fields are generated by source functions of the form

$$(1.11) \qquad f(t,q) = g_1(q) \cos \omega t + g_2(q) \sin \omega t = \text{Re } \{g(q) \, e^{-i\omega t}\}$$

where $\omega \geq 0$ and $g = g_1 + ig_2$. The corresponding steady-state acoustic potentials have the same form

$$(1.12) \qquad u(t,q) = v_1(q) \cos \omega t + v_2(q) \sin \omega t = \text{Re } \{v(q) \, e^{-i\omega t}\}$$

where $v = v_1 + iv_2$. $u(t,q)$ satisfies (1.7) and (1.9) or (1.10) for all $t \in R$ if and only if

$$(1.13) \qquad\qquad \Delta v + \omega^2 v = -g(q), \; q \in \Omega$$

and

$$(1.14) \qquad\qquad v\Big|_{\partial\Omega} = 0 \text{ or } \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

4

These conditions do not determine v uniquely. Instead, solutions may differ by any solution of the reduced problem with $g(q) \equiv 0$ in Ω. Among these are the solutions in $L_2(\Omega)$, corresponding to eigenvalues ω^2 of the operator $-\Delta$ acting in $L_2(\Omega)$. F. Rellich [18] and D. S. Jones [10] have shown that such eigenvalues may exist when $\Gamma'' = 0$ (no conical horns in Ω). If $\partial\Omega$ has sharp edges or vertices the reduced problem may have "edge wave" solutions which correspond physically to sources hidden in the edge. For example if $\Omega = \{q \in E^3 : 0 < \theta(q) < \pi/2\}$ then

$$(1.15) \qquad\qquad\qquad v = H_0^{(1)}(\omega\rho)$$

satisfies (1.13) with $g(q) \equiv 0$ in Ω and the Neumann condition on $\partial\Omega$. Edge waves are eliminated by requiring that the time-averaged energy of the field be locally finite:

$$(1.16) \qquad \int_{K \cap \Omega} \{|\nabla v|^2 + \omega^2 |v|^2\} \, dV < \infty \text{ for every compact } K \subset E^3$$

This condition is called the "edge condition" in the older literature [2, 14].

The physically important steady-state fields behave like outgoing waves in each unbounded component of Ω. These solutions are characterized by a radiation condition of the Sommerfeld type which eliminates another type of non-uniqueness caused by "incoming wave" terms in $v(q)$.

The paper is organized as follows. In section 2 selfadjoint operators Δ_Ω^D and Δ_Ω^N in $L_2(\Omega)$ are associated with the Laplacian and the Dirichlet and Neumann conditions, respectively. In section 3 the radiation condition is defined and used to formulate the uniqueness theorem. The existence of outgoing steady-state fields with prescribed source functions g is proved in section 4 as a corollary of a limiting absorption theorem. The latter,

which is the central analytical result of the paper, asserts that the resolvent $R_\Omega^D(z) = -(\Delta_\Omega^D + z)^{-1}$ (resp. $R_\Omega^N(z) = -(\Delta_\Omega^N + z)^{-1}$) has limits when $z = \omega^2 \pm i\sigma \to \omega^2 \pm i0$ provided that ω^2 is not an eigenvalue of $-\Delta_\Omega^D$ (resp. $-\Delta_\Omega^N$) and ω is not a cutoff frequency (see section 3) of one of the waveguides S_i or S_j'.

Section 5 contains a discussion of the proofs of the theorems formulated in sections 2-4. These theorems are generalizations to waveguide domains of results of Wilcox on scattering in exterior domains. The latter were presented in the monograph [23] to which reference is made for details of the proofs. The discussion in section 5 is limited to those points in which the proofs of [23] must be modified to treat waveguides.

Related Literature. The principle of limiting absorption has a long history. An early statement of it was given in 1905 by W. Ignatowsky [9] and it has often been used as a heuristic principle in the physical literature. It was proved for the waveguides formed by parallel plates and perfect cylinders by A. G. Svešnikov [20, 21]. The first general proof is due to D. M. Eidus [4] who studied elliptic operators with variable coefficients in exterior domains and perturbed cylinders. C. I. Goldstein has also proved the principle for perturbations of cylinders as part of his spectral and scattering theory of waveguides [5, 6, 7]. The case of finite perturbations of a cone in E^2 was treated by Goldstein in [5, III]. Generalizations to higher-dimensional cones and more general differential operators and boundary conditions were announced but not proved in that paper. W. C. Lyford [12] has proved the limiting absorption principle for compound cylindrical waveguides in the same generality as here. The authors know of no treatment by the methods of this paper of compound waveguides containing sectoral and conical horns. Most of the results of this paper were announced by the authors in [8].

8

(2.6) $(u, \nabla \cdot \vec{V}) + (\nabla u, \vec{V}) = 0$ for all $\vec{V} \in L_2(\nabla \cdot, \Omega, C^3)$

and the generalized Neumann condition

(2.7) $(\Delta u, v) + (\nabla u, \nabla v) = 0$ for all $v \in L_2^1(\Omega)$

respectively. These conditions are applicable to arbitrary domains Ω and are equivalent to the classical conditions when $\partial \Omega$ and u are sufficiently regular [22, 23]. Operators Δ_Ω^D and Δ_Ω^N in $L_2(\Omega)$ are defined by Δ acting on the domains

(2.8) $D(\Delta_\Omega^D) = L_2^1(\Delta, \Omega) \cap \{u: \ u \ \text{satisfies} \ (2.6)\}$

(2.9) $D(\Delta_\Omega^N) = L_2^1(\Delta, \Omega) \cap \{u: \ u \ \text{satisfies} \ (2.7)\}$

They provide the starting point for the theory of acoustic waveguides developed below. Their utility is due to [12, 23]

<u>Theorem 1</u>. For every domain $\Omega \subset E^3$ the operators Δ_Ω^D and Δ_Ω^N are selfadjoint non-positive operators in $L_2(\Omega)$.

The theorems in the remainder of the paper are based on a local compactness property of operators in $L_2(\Omega)$ described by the

<u>Definition</u>. A linear operator A in $L_2(\Omega)$ with domain $D(A) \subset L_2^1(\Omega)$ is said to have the local compactness property (in symbols, $A \in LC$) if and only if for every bounded measurable set $K \subset E^3$ and subset $S \subset D(A)$ which is bounded in $L_2^1(K \cap \Omega)$ the set S is precompact in $L_2(K \cap \Omega)$.

The embedding of $L_2^1(\Omega)$ in $L_2^1(K \cap \Omega)$ and $L_2(K \cap \Omega)$ implied by the definition is the natural embedding which assigns to any function on Ω its restriction to $K \cap \Omega$.

A domain Ω is said to have the Rellich property if, for each bounded measurable $K \subset E^3$, the natural embedding of $L_2^1(K \cap \Omega)$ into $L_2(K \cap \Omega)$ is

compact. If this property holds then every operator A with $D(A) \subset L_2^1(\Omega)$ satisfies $A \in$ LC. Not all domains have the Rellich property [3]. The property was proved by Rellich [17] for domains with piecewise smooth boundaries and by S. Agmon for domains having the "segment property" [1, Theorem 3.8]. Wilcox [23] extended Agmon's proof to domains with the "finite tiling property." Thus $\Delta_\Omega^N \in$ LC for these classes. Of course $\Delta_\Omega^D \in$ LC for the same class. However, in the case of the Dirichlet condition restrictions on $\partial\Omega$ are not needed. Instead, one has

$$(2.10) \qquad \Delta_\Omega^D \in \text{LC for every domain } \Omega \subset E^3$$

A proof is indicated in section 5.

In the remainder of the paper the class of domains Ω defined by (1.1)-(1.6) is denoted by W^D and the subset consisting of domains $\Omega \in W^D$ such that $\Delta_\Omega^N \in$ LC is denoted by W^N. Moreover, in discussions of Δ_Ω^D (resp. Δ_Ω^N) it is always assumed that $\Omega \in W^D$ (resp. W^N). With this understanding the theories of Δ_Ω^D and Δ_Ω^N are identical. For conciseness the symbol Δ_Ω is used below to denote either Δ_Ω^D or Δ_Ω^N in statements that are valid for both.

The study of the point spectrum of Δ_Ω was begun by F. Rellich [18] and extended by D. S. Jones [10]. They showed that domains Ω having a conical horn (I" $\geq$ 1) have no point spectrum, while domains with no conical horn (I" = 0) may have point spectrum. Geometric criteria for its existence were given and Jones showed that it must be discrete; i.e., each interval can have only finitely many eigenvalues and each has finite multiplicity. The work of Rellich and Jones involved stronger local regularity hypotheses on $\partial\Omega$ than are made here. However, their methods of proof are based on the quadratic forms associated with Δ_Ω and can be extended to all domains for which $\Delta_\Omega \in$ LC.

SECTION THREE

THE UNIQUENESS OF STEADY-STATE WAVE FIELDS

The asymptotic behavior of steady-state fields in the unbounded cylinder and/or horn portions of Ω is determined in this section and used to formulate a radiation condition and corresponding uniqueness theorem.

The basic space of this section is defined by

$$(3.1) \qquad L_2^{loc}(\overline{\Omega}) = \mathcal{D}'(\Omega) \cap \{u: \ u \in L_2(K \cap \Omega) \text{ for every compact } K \subset E^3\}$$

where $\mathcal{D}'(\Omega)$ is L. Schwartz's space of distributions in Ω. $L_2^{loc}(\overline{\Omega})$ must be distinguished from the more familiar space

$$(3.2) \qquad L_2^{loc}(\Omega) = \mathcal{D}'(\Omega) \cap \{u: \ u \in L_2(K) \text{ for every compact } K \subset \Omega\}$$

Note that the condition $u \in L_2^{loc}(\overline{\Omega})$ restricts u near $\partial\Omega$ while $u \in L_2^{loc}(\Omega)$ does not. The local analogues of the remaining spaces of section 2 are also needed. They are

$$(3.3) \qquad L_2^{loc}(\overline{\Omega},C^3) = \mathcal{D}'(\Omega,C^3) \cap \{\vec{A}: \ \vec{A} \in L_2(K \cap \Omega,C^3) \text{ for every compact } K \subset E^3\}$$

$$(3.4) \qquad L_2^{1,loc}(\overline{\Omega}) = L_2^{loc}(\overline{\Omega}) \cap \{u: \ \nabla u \in L_2^{loc}(\overline{\Omega},C^3)\}$$

$$(3.5) \qquad L_2^{1,loc}(\Delta,\overline{\Omega}) = L_2^{1,loc}(\overline{\Omega}) \cap \{u: \ \Delta u \in L_2^{loc}(\overline{\Omega})\}$$

$$(3.6) \qquad L_2^{loc}(\nabla\cdot,\overline{\Omega},C^3) = L_2^{loc}(\overline{\Omega},C^3) \cap \{\vec{A}: \ \nabla\cdot\vec{A} \in L_2^{loc}(\overline{\Omega})\}$$

The following spaces are also needed

$$(3.7) \qquad L_2^{com}(\overline{\Omega}) = E'(E^3) \cap L_2(\Omega)$$

12

$$(3.8) \qquad L_2^{1,com}(\overline{\Omega}) = L_2^{com}(\overline{\Omega}) \cap L_2^1(\Omega)$$

$$(3.9) \qquad L_2^{com}(\overline{\Omega}, C^3) = E'(E^3, C^3) \cap L_2(\Omega, C^3)$$

$$(3.10) \qquad L_2^{com}(\nabla \cdot, \overline{\Omega}, C^3) = L_2^{com}(\overline{\Omega}, C^3) \cap L_2(\nabla \cdot, \Omega, C^3)$$

where $E'(E^3)$ is L. Schwartz's space of distributions in E^3 with compact support.

Operators $\Delta_\Omega^{D,loc}$ and $\Delta_\Omega^{N,loc}$ in $L_2^{loc}(\overline{\Omega})$ are defined by Δ acting on the domains

$$(3.11) \quad D(\Delta_\Omega^{D,loc}) = L_2^{1,loc}(\Delta, \overline{\Omega}) \cap \{u: (2.6) \text{ holds for all } \vec{v} \in L_2^{com}(\nabla \cdot, \overline{\Omega}, C^3)\}$$

$$(3.12) \quad D(\Delta_\Omega^{N,loc}) = L_2^{1,loc}(\Delta, \overline{\Omega}) \cap \{u: (2.7) \text{ holds for all } v \in L_2^{1,com}(\overline{\Omega})\}$$

In section 4 it is shown that steady-state fields with source functions $g \in L_2^{com}(\overline{\Omega})$ exist in the domains (3.11), (3.12). Such fields satisfy the edge condition (1.15) and the generalized Dirichlet or Neumann condition. For brevity, the notation Δ_Ω^{loc} will be used to denote either $\Delta_\Omega^{D,loc}$ or $\Delta_\Omega^{N,loc}$ in statements which are valid for both.

Let $v \in D(\Delta_\Omega^{loc})$ satisfy

$$(3.13) \qquad \Delta v + \omega^2 v = -g(q) \in L_2^{com}(\overline{\Omega})$$

where $\omega \geq 0$. To study the asymptotic behavior of $v(q)$ it will be convenient to restrict it to the sets S_i, S_j', S_k'' by means of a smooth cutoff function. To do this choose a function $j \in C^\infty(R)$ such that

$$(3.14) \qquad j(\tau) \equiv 0 \text{ for } \tau \leq 0, \ j(\tau) \equiv 1 \text{ for } \tau \geq 1, \ 0 \leq j(\tau) \leq 1$$

and associate with each function u on Ω the functions u_i on S_i, u_j' on S_j', u_k'' on S_k'' defined by

(3.15) $u_i(q) = j(y^i(q) - y_0^i - 1)u(q)$ for all $q \in S_i$

(3.16) $u_j^!(q) = j(\rho^j(q) - \rho_0^j - 1)u(q)$ for all $q \in S_j^!$

(3.17) $u_k^{''}(q) = j(r^k(q) - r_0^k - 1)u(q)$ for all $q \in S_k^{''}$

The corresponding linear operators, defined by

(3.18) $J_i u = u_i, \quad J_j^! u = u_j^!, \quad J_k^{''} u = u_k^{''}$

have the properties described by (cf. [12])

Proposition 3.1. The restrictions of J_i, $J_j^!$, $J_k^{''}$ to the spaces $L_2^{loc}(\overline{\Omega})$, $D(\Delta_\Omega^{loc})$, $L_2(\Omega)$ and $D(\Delta_\Omega)$ are linear continuous mappings of these spaces into the corresponding spaces for S_i, $S_j^!$ and $S_k^{''}$ respectively.

The topologies of the spaces $L_2^{loc}(\overline{\Omega})$, $D(\Delta_\Omega^{loc})$ referred to in Proposition 3.1 are the Fréchet topologies defined in [12, 23]. The structure of the functions v_i, $v_j^!$, $v_k^{''}$ corresponding to (3.13) will be analyzed by means of the special coordinate systems that define S_i, $S_j^!$, $S_k^{''}$ and spectral analysis with respect to the Laplace-Beltrami operators for $G_i \subset R^2$, $G_j^! \subset S^1 \times R$, $G_k^{''} \subset S^2$. The corresponding selfadjoint non-positive operators in $L_2(G_i)$, $L_2(G_j^!)$, $L_2(G_k^{''})$ will be denoted by Δ_i^D and Δ_i^N, $\Delta_j^{!D}$ and $\Delta_j^{!N}$, $\Delta_k^{''D}$ and $\Delta_k^{''N}$, respectively. Their definitions are entirely analogous to those of Δ_Ω^D and Δ_Ω^N in section 2. In fact, the definitions given there are applicable to the Laplace-Beltrami operators in subdomains of any Riemannian manifold. The notations Δ_i, $\Delta_j^!$, $\Delta_k^{''}$ will be used to denote either the Dirichlet or the Neumann operators in statements that are valid for both. The discreteness of the spectra of these operators follows from the following proposition whose proof is indicated in section 5.

__Proposition 3.2__. The hypothesis $\Omega \in W^D$ (resp., $\Omega \in W^N$) implies that Δ_i^D and $\Delta_k^{''D}$ (resp., Δ_i^N and $\Delta_k^{''N}$) have the local compactness property.

Note that $\Delta_j^{'D} \in LC$ and $\Delta_j^{'N} \in LC$ because $G_j^{'}$ is just a rectangle in $S^1 \times R$, a domain with the Rellich property. These observations and the boundedness of G_i, $G_j^{'}$, $G_k^{''}$ imply that $-\Delta_i$, $-\Delta_j^{'}$, $-\Delta_k^{''}$ have discrete spectra lying in $[0, \infty)$. The eigenvalues, enumerated in increasing order and repeated according to multiplicity, will be denoted by $\{\omega_{i\ell}^2\}$, $\{\mu_{j\ell}^2\}$, $\{\nu_{k\ell}^2\}$ respectively where $\ell = 1, 2, 3, \cdots$. Corresponding complete orthonormal sets of eigenfunctions in $L_2(G_i)$, $L_2(G_j^{'})$, $L_2(G_k^{''})$ will be denoted by $\{\phi_{i\ell}\}$, $\{\phi_{j\ell}^{'}\}$, $\{\phi_{k\ell}^{''}\}$. The sequences $\{\mu_{j\ell}^2\}$, $\{\phi_{j\ell}^{'}\}$ will be constructed from those for the operators $-\Delta_{M_j^{'}} = -d^2/d\theta^2$ in $L_2(M_j^{'})$ and $-\Delta_{(0, z_0^j)} = -d^2/dz^2$ in $L_2(0, z_0^j)$ by separation of variables; i.e.,

$$(3.19) \qquad \mu_{j\ell}^2 = \nu_{jm(\ell)}^{'2} + \omega_{jn(\ell)}^{'2}$$

$$(3.20) \qquad \phi_{j\ell}^{'}(\theta, z) = a_{jm(\ell)}(\theta) \, b_{jn(\ell)}(z)$$

where $(d^2/d\theta^2 + \nu_{jm}^{'2})a_{jm} = 0$, $(d^2/dz^2 + \omega_{jn}^{'2})b_{jn} = 0$ and $\ell \leftrightarrow (m(\ell), n(\ell))$ is a suitable enumeration of the pairs (m, n) of positive integers.

It will be convenient to represent u_i, $u_j^{'}$, $u_k^{''}$ as functions of the special coordinates in S_i, $S_j^{'}$, $S_k^{''}$. The following notation will be used

$$(3.21) \qquad u_i(x^i, y^i) = u_i(q) \text{ where } (x^i, y^i) \leftrightarrow q \in S_i$$

$$(3.22) \qquad u_j^{'}(\rho^j, \theta^j, z^j) = u_j^{'}(q) \text{ where } (\rho^j, \theta^j, z^j) \leftrightarrow q \in S_j^{'}$$

$$(3.23) \qquad u_k^{''}(r^k, \eta^k) = u_k^{''}(q) \text{ where } (r^k, \eta^k) \leftrightarrow q \in S_k^{''}$$

The expandability of these functions in the eigenfunctions of Δ_i, $\Delta_j^{'}$, $\Delta_k^{''}$ whenever $u \in D(\Delta_\Omega^{loc})$ follows from [22, Theorem 2.2]

Proposition 3.3. The correspondences (3.21)-(3.23) define continuous embeddings

$$(3.24) \qquad D(\Delta_{S_i}^{loc}) \subset C([y_0^i, \infty), L_2(G_i))$$

$$(3.25) \qquad D(\Delta_{S_j'}^{loc}) \subset C([\rho_0^j, \infty), L_2(G_j'))$$

$$(3.26) \qquad D(\Delta_{S_k''}^{loc}) \subset C([r_0^k, \infty), L_2(G_k''))$$

Propositions 3.1 and 3.3 imply that for every $u \in D(\Delta_\Omega^{loc})$ the coefficient functions

$$(3.27) \qquad u_{i\ell}(y) = \int_{G_i} \overline{\phi_{i\ell}(x)} \, u_i(x,y) \, dx$$

$$(3.28) \qquad u_{j\ell}'(\rho) = \int_{G_j'} \overline{\phi_{j\ell}'(\theta,z)} \, u_j'(\rho,\theta,z) \, d\theta dz$$

$$(3.29) \qquad u_{k\ell}''(r) = \int_{G_k''} \overline{\phi_k''(\eta)} \, u_k''(r,\eta) \, d\eta$$

are continuous and one has the following two propositions.

Proposition 3.4. For every $u \in D(\Delta_\Omega^{loc})$ the following expansions are valid.

$$(3.30) \qquad u_i(x,y) = \sum_{\ell=1}^\infty u_{i\ell}(y) \, \phi_{i\ell}(x), \text{ convergent in } D(\Delta_{S_i}^{loc})$$

$$(3.31) \qquad u_j'(\rho,\theta,z) = \sum_{\ell=1}^\infty u_{j\ell}'(\rho) \, \phi_{j\ell}'(\theta,z), \text{ convergent in } D(\Delta_{S_j'}^{loc})$$

$$(3.32) \qquad u_k''(r,\eta) = \sum_{\ell=1}^\infty u_{k\ell}''(r) \, \phi_{k\ell}''(\eta), \text{ convergent in } D(\Delta_{S_k''}^{loc})$$

16

Proposition 3.5. Let $v \in D(\Delta_\Omega^{loc})$ satisfy (3.13) and let $g_i(x,y) = 0$
for $y > y^i(g)$, $g_j'(\rho,\theta,z) = 0$ for $\rho > \rho^j(g)$ and $g_k''(r,\eta) = 0$ for $r > r^k(g)$.
Then the coefficients $v_{i\ell}$, $v_{j\ell}'$, $v_{k\ell}''$ satisfy the differential equations

$$(3.33) \qquad L\left[\frac{d}{dy}, \omega^2 - \omega_{i\ell}^2, 1/2\right] v_{i\ell}(y) = 0 \text{ for } y > y^i(g)$$

$$(3.34) \qquad L\left[\frac{d}{d\rho}, \omega^2 - \omega_{jn(\ell)}'^2, \nu_{jm(\ell)}'\right] \rho^{1/2} v_{j\ell}'(\rho) = 0 \text{ for } \rho > \rho^j(q)$$

$$(3.35) \qquad L\left[\frac{d}{dr}, \omega^2, (\nu_{k\ell}^2 + 1/4)^{1/2}\right] r v_{k\ell}''(r) = 0 \text{ for } r > r^k(g)$$

where

$$(3.36) \qquad L\left[\frac{d}{dx}, a, b\right] = \frac{d^2}{dx^2} + (a - (b^2 - 1/4)/x^2)$$

A solution basis for (3.33) is defined by

$$(3.37) \qquad \exp\{i(\omega^2 - \omega_{i\ell}^2)^{1/2} y\}, \ \exp\{-i(\omega^2 - \omega_{i\ell}^2)^{1/2} y\}, \ \omega \neq \omega_{i\ell}$$

$$(3.38) \qquad 1, \quad y \qquad\qquad\qquad , \ \omega \approx \omega_{i\ell}$$

Solution bases for (3.35), (3.36) may be based on the usual Hankel functions
$H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ which have the asymptotic forms [13]

$$(3.39) \qquad H_\nu^{(1,2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{\pm i(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)}, \ z \to \infty$$

where $1 \Leftrightarrow +$ and $2 \Leftrightarrow -$. With this notation a solution basis for (3.34) is

$$(3.40) \qquad H_\nu^{(1,2)}((\omega^2 - \omega_{jn(\ell)}'^2)^{1/2} \rho), \ \nu = \nu_{jm(\ell)}', \ \omega \neq \omega_{jn(\ell)}'$$

$$(3.41) \qquad \rho^{-\nu_{jm(\ell)}'}, \ \rho^{\nu_{jm(\ell)}'}, \ \omega = \omega_{jn(\ell)}', \ \nu_{jm(\ell)}' \neq 0$$

(3.42) $1,$ $\ln \rho$ $, \omega = \omega'_{jn(\ell)}, \; \nu'_{jm(\ell)} = 0$

Similarly, a solution basis for (3.35) is

$$(3.43) \qquad r^{-1/2} \; H_\nu^{(1,2)}(\omega r), \; \nu = (\nu_{k\ell}^2 + \tfrac{1}{4})^{1/2}, \; \omega \neq 0$$

$$(3.44) \qquad r^{-\frac{1}{2}-(\nu_{k\ell}^2+\frac{1}{4})^{1/2}}, \quad r^{-\frac{1}{2}+(\nu_{k\ell}^2+\frac{1}{4})^{1/2}}, \; \omega = 0$$

When multiplied by the time-factor $e^{-i\omega t}$ of (1.11) the solutions (3.37) describe propagating waves in S_i if and only if $\omega > \omega_{i\ell}$. Similarly, (3.40) and (3.43) define propagating waves in S'_j and S''_k if and only if $\omega > \omega'_{jn(\ell)}$ and $\omega > 0$, respectively. The numbers $\{\omega_{i\ell}\}$, $\{\omega'_{jn}\}$, 0 are called the "cutoff frequencies" for the waveguides S_i, S'_j, S''_k, respectively. The notation

$$(3.45) \qquad \sigma_{\text{cutoff}}(\Omega) = \left\{ \bigcup_{i=1}^{I} \bigcup_{\ell=1}^{\infty} \{\omega_{i\ell}\} \right\} \cup \left\{ \bigcup_{j=1}^{I'} \bigcup_{n=1}^{\infty} \{\omega'_{jn}\} \right\} \cup \{0\}$$

will be used.

Concerning the point spectrum of Δ_Ω, it is clear from Propositions 3.4 and 3.5 that every eigenfunction must have coefficients v_i, v'_j, v''_k which are zero in all propagating modes and decrease exponentially in all cutoff modes. In particular, if Ω contains a conical horn ($I'' \geq 1$) then Δ_Ω can have no point spectrum, as was proved by Rellich.

The goal of this paper is to show that for all frequencies $\omega \geq 0$ except cutoff frequencies and the square roots of eigenvalues of $-\Delta_\Omega$ there exists in $D(\Delta_\Omega^{\ell oc})$ a unique solution of (3.13) which is a pure outgoing wave in each propagating mode and is exponentially decreasing in each cutoff mode. The description of these solutions is facilitated by the use of the functions $(\lambda \pm io)^{1/2}$ defined by

18

$$(3.46) \qquad \text{Im } z^{1/2} > 0 \text{ for all } z \in C - [0,\infty)$$

$$(3.47) \qquad (\lambda \pm io)^{1/2} = \lim_{\epsilon \to +o} (\lambda \pm i\epsilon)^{1/2}$$

Note that

$$(3.48) \qquad \pm(\lambda \pm io)^{1/2} = \lambda^{1/2} \geq 0 \text{ for all } \lambda \geq 0$$

$$(3.49) \qquad -i (\lambda \pm io)^{1/2} = (-\lambda)^{1/2} > 0 \text{ for all } \lambda < 0$$

<u>Definition</u>. Let $\omega \in [0,\infty) - \sigma_{cutoff}(\Omega)$. Then a solution $v \in D(\Delta_{\Omega}^{loc})$ of (3.13) is said to satisfy the outgoing radiation condition if and only if there exist constants $c_{i\ell}(\omega)$, $c'_{j\ell}(\omega)$, $c''_{k\ell}(\omega)$ such that

$$(3.50) \qquad v_{i\ell}(y) = c_{i\ell}(\omega) \exp \{i(\omega^2 - \omega_{i\ell}^2 + io)^{1/2} y\} \quad \text{for } y > y^i(g)$$

$$(3.51) \qquad \begin{aligned} v'_{j\ell}(\rho) &= c'_{j\ell}(\omega) H_\nu^{(1)}((\omega^2 - \omega_{jn(\ell)}'^2 + io)^{1/2} \rho), \\ &\qquad\qquad \nu = \nu'_{jm(\ell)} \text{ for } \rho > \rho^j(g) \end{aligned}$$

$$(3.52) \qquad \begin{aligned} v''_{k\ell}(r) &= c''_{k\ell}(\omega) \, r^{-1/2} H_\nu^{(1)}((\omega^2 + io)^{1/2} r), \\ &\qquad\qquad \nu = (\nu_{k\ell}^2 + \tfrac{1}{4})^{1/2} \text{ for } r > r^k(g) \end{aligned}$$

v is said to satisfy the incoming radiation condition if (3.50)-(3.52) hold with $(\lambda + io)^{1/2}$ replaced by $(\lambda - io)^{1/2}$.

The uniqueness theorem can now be formulated as follows.

<u>Theorem 2</u>. Let $\omega \in [0,\infty) - \sigma_{cutoff}(\Omega)$ and let $v \in D(\Delta_{\Omega}^{loc})$ satisfy

$$(3.53) \qquad \Delta v + \omega^2 v = 0 \text{ in } \Omega$$

(3.54) $\quad v$ satisfies the outgoing or incoming radiation condition

(3.55)   v is orthogonal in $L_2(\Omega)$ to the eigenspace of $-\Delta_\Omega$ for the eigen-
value ω^2 (which contains only the zero vector if ω^2 is not an
eigenvalue)

Then $v(q) = 0$ in Ω. Moreover, if Ω has a conical horn component $(I'' \geq 1)$
the same result holds for all $\omega \geq 0$.

Note that condition (3.55) is meaningful, even though v is not in
$L_2(\Omega)$, because the eigenfunctions of Δ_Ω are exponentially decreasing in each
waveguide and (3.54) implies that the coefficients v_i, v_j', v_k'' are bounded.

Theorem 2 clearly implies the uniqueness of solutions in $D(\Delta_\Omega^{loc})$ of
(3.13) which satisfy (3.54), (3.55). Of course, this result is of interest
only if such solutions exist. In section 4 Theorem 2 is shown to imply the
validity of a limiting absorption principle and, as a consequence, the
existence of outgoing and incoming solutions in $D(\Delta_\Omega^{loc})$ for every $g \in L_2^{com}(\overline{\Omega})$.

SECTION FOUR

THE LIMITING ABSORPTION PRINCIPLE

The spectrum, the point spectrum and the continuous spectrum of the operator $-\Delta_\Omega$ will be denoted by $\sigma(\Omega)$, $\sigma_p(\Omega)$ and $\sigma_c(\Omega)$, respectively. Thus $\sigma(\Omega) \subset [0,\infty)$ and $\sigma(\Omega) = \sigma_p(\Omega) \cup \sigma_c(\Omega)$ [11]. The resolvent operator of $-\Delta_\Omega$,

$$(4.1) \qquad R_\Omega(z) = -(\Delta_\Omega + z)^{-1}, \ z \in C - \sigma(\Omega)$$

is bounded in $L_2(\Omega)$ and $z \to R_\Omega(z)$ defines a holomorphic $L_2(\Omega)$ operator-valued function in $C - \sigma(\Omega)$ [11]. This function cannot be extended to points $z \in \sigma(\Omega)$. However, it will be shown that if $g \in L_2^{com}(\overline{\Omega})$ then the mapping $z \to R_\Omega(z)g \in D(\Delta_\Omega^{loc})$ has one-sided limits at the points of the set

$$(4.2) \qquad \Lambda = \sigma_c(\Omega) - (\sigma_{cutoff}(\Omega))^2$$

This result is a corollary of

Theorem 3. For each $g \in L_2^{com}(\overline{\Omega})$ the mappings T_+ and T_- from $\{(\lambda,\sigma): \ \lambda \geq 0$ and $\sigma > 0\}$ to $D(\Delta_\Omega^{loc})$ defined by

$$(4.3) \qquad T_\pm(\lambda,\sigma) = R_\Omega(\lambda \pm i\sigma)g \in D(\Delta_\Omega^{loc})$$

are continuous. Moreover, for each interval $I = [a,b] \subset \Lambda$ and each $\sigma_0 > 0$, T_+ and T_- are uniformly continuous on $I \times (0,\sigma_0]$.

This result was proved in [23] for the special case of exterior domains $(G_k'' = S^2)$. The extension of the proof to the class of waveguide domains W^D and W^N is given in section 5. An immediate corollary of the uniform continuity statement of Theorem 3 is the

22

Limiting Absorption Principle. The mappings T_+ and T_- have continuous extensions to the set

$$(4.4) \qquad \{(\lambda,\sigma): \ \lambda \geq 0, \sigma > 0\} \cup \{\Lambda \times [0,\infty)\}$$

Moreover, the limits $T_\pm(\lambda,0+) = R_\Omega(\lambda \pm io)g$, $\lambda \in \Lambda$, define operators $R_\Omega(\lambda \pm io)$ from $L_2^{com}(\overline{\Omega})$ to $D(\Delta_\Omega^{loc})$ which are bounded uniformly on compact subsets of Λ.

A second corollary is the following existence theorem for steady-state fields in waveguides.

Theorem 4. For each $\omega \geq 0$ such that $\omega^2 \in \Lambda$ and each $g \in L_2^{com}(\overline{\Omega})$ let

$$(4.5) \qquad v^\pm(\cdot,\omega) = R_\Omega(\omega^2 \pm io)g \in D(\Delta_\Omega^{loc})$$

Then $v^+(q,\omega)$ (resp. $v^-(q,\omega)$) is the (unique) outgoing (resp. incoming) solution in $D(\Delta_\Omega^{loc})$ of (3.13).

SECTION FIVE

PROOFS OF THE THEOREMS

The validity of $\Delta_\Omega^D \in LC$ for every $\Omega \subset E^3$ follows from the characterization

(5.1) $$D(\Delta_\Omega^D) = L_2^{1,0}(\Omega) \cap L_2^1(\Delta,\Omega),$$

where $L_2^{1,0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $L_2^1(\Omega)$, and from the fact that for bounded domains Ω' the natural embedding of $L_2^{1,0}(\Omega')$ into $L_2(\Omega')$ is compact [1, Theorem 8.3]. To prove (5.1) note that $L_2^{1,0}(\Omega) \cap L_2^1(\Delta,\Omega) = D(A)$ where A is the selfadjoint operator associated with the sesquilinear form $A(u,v) = (\nabla u, \nabla v)$ with domain $D(A) = L_2^{1,0}(\Omega)$ [11, p. 322]. It is obvious that $A \subset -\Delta_\Omega^D$. Hence $A = -\Delta_\Omega^D$ by the maximality of selfadjoint operators which proves (5.1).

Proof of Proposition 3.2. A proof will be indicated for the operators Δ_i corresponding to the cylinders S_i. The other cases can be proved similarly. The property $\Delta_i^D \in LC$ follows by the method of the preceding paragraph. To prove $\Delta_i^N \in LC$ let $S = \{v(x)\} \subset D(\Delta_i^N)$ be bounded in $L_2^1(G_i)$. Let $\chi_i(q)$ be the characteristic function of S_i, $\phi \in C_0^\infty(R)$, $\phi(y) \equiv 1$ for $y_0^i + 1 \leq y \leq y_0^i + 2$, $\phi(y) \equiv 0$ outside $y_0^i + \frac{1}{2} \leq y \leq y_0^i + \frac{5}{2}$ and define $S' = \{u: u(q) = v(x^i(q)) \phi(y^i(q)) \chi_i(q), v \in S\}$. Also let $K_i = \{q \in E^3: x^i(q) \in G_i$ and $y_0^i + 1 < y^i(q) < y_0^i + 2\}$. Then K_i is bounded and measurable and $K_i \cap \Omega = K_i$. Moreover, $\|u\|_{L_2^1(K_i)} = \|v\|_{L_2^1(G_i)} \leq C$ for all $v \in S$. Since $\|u\|_{L_2(K_i)} = \|v\|_{L_2(G_i)}$ for all $v \in S$ the hypothesis $\Omega \in V^N$ implies that S is compact in $L_2(G_i)$.

<u>Proof of Theorem 2</u>. Note that (3.53) implies that (3.50)-(3.52) hold with $y^i(g) = y_0^i$, $\rho^j(g) = \rho_0^j$, $r^k(g) = r_0^k$. Choose a number $R > \text{Max } (y_0^i, \rho_0^j, r_0^k)$ and define (cf. (1.2)-(1.4))

$$(5.2) \qquad \Omega_R = \Omega - \bigcup_{i=1}^{I} S_i(R) - \bigcup_{j=1}^{I'} S_j'(R) - \bigcup_{k=1}^{I''} S_k''(R)$$

Now apply Green's theorem to v and $\overline{v}$ in Ω_R. This must be done by applying the generalized boundary condition (2.6) (resp. (2.7)) with $u \to v$ and $\vec{V} \to \chi\overline{\nabla v}$ (resp. $u \to v$ and $v \to \chi\overline{v}$) where χ is a smooth cutoff function. The technique is explained in [23, p. 57ff]. The result is, by (3.53),

$$0 = \int_{\Omega_R} \{v\Delta\overline{v} - \overline{v}\Delta v\} \, dV$$

$$
\begin{aligned}
(5.3) \qquad = 2\,i\,\text{Im} \Bigg\{ &\sum_{i=1}^{I} \int_{G_i} \left< \left\{ v\,\frac{\partial \overline{v}}{\partial y} \right\} \right>_{y=R} dx + \sum_{j=1}^{I'} \int_{G_j'} \left< \left\{ v\,\frac{\partial \overline{v}}{\partial \rho} \right\} \right>_{\rho=R} R\,d\theta\,dz \\
&+ \sum_{k=1}^{I''} \int_{G_k''} \left< \left\{ v\,\frac{\partial \overline{v}}{\partial r} \right\} \right>_{r=R} R^2\,d\eta \Bigg\}
\end{aligned}
$$

Applying Proposition 3.4 to v and using Parseval's relation in $L_2(G_i)$, $L_2(G_j')$ and $L_2(G_k'')$ gives

$$
\begin{aligned}
0 = \text{Im} \Bigg\{ &\sum_{i=1}^{I} \sum_{\ell=1}^{\infty} v_{i\ell}(R)\,\frac{d\overline{v}_{i\ell}(R)}{dR} + \sum_{j=1}^{I'} \sum_{\ell=1}^{\infty} v_{j\ell}'(R)\,\frac{d\overline{v}_{j\ell}'(R)}{dR} \\
(5.4) \\
&+ \sum_{k=1}^{I''} \sum_{\ell=1}^{\infty} v_{k\ell}''(R)\,\frac{d\overline{v}_{k\ell}''(R)}{dR} \Bigg\}
\end{aligned}
$$

Finally, using (3.50)-(3.52) and the Wronskians for equations (3.33)-(3.35) gives, after simplification,

$$\frac{\pi}{2} \sum_{i=1}^{I} \sum_{\omega > \omega_{i\ell}} (\omega^2 - \omega_{i\ell}^2 + i0)^{1/2} |c_{i\ell}(\omega)|^2 + \sum_{j=1}^{I'} \sum_{\omega > \omega'_{jn(\ell)}} |c'_{j\ell}(\omega)|^2$$

(5.5)

$$+ \sum_{k=1}^{I''} \sum_{\ell=1}^{\infty} |c''_{k\ell}(\omega)|^2 = 0$$

All the terms in these sums are non-negative; see (3.48). Hence, the sums (3.30)-(3.32) for v are finite and have only exponentially decreasing terms since $\omega \in [0,\infty) - \sigma_{cutoff}(\Omega)$. Thus $v \in L_2(\Omega)$ and therefore $v \in D(\Delta_\Omega)$ and $-\Delta_\Omega v = \omega^2 v$ which, with (3.55), implies $v = 0$. Note that if $\omega \in \sigma_{cutoff}(\Omega)$ then one cannot conclude that $v \in L_2(\Omega)$ because of the exceptional solutions (3.38), (3.41), (3.42), (3.44). However, if $I'' \geq 1$ then all $c''_{k\ell}(\omega) = 0$, as before, and hence $v(q) = 0$ in S''_k. The analyticity of solutions of (3.53) then implies that $v(q) = 0$ in Ω.

Proof of Theorem 3. The proof is the same, except for one step, as the proof given in [23, pp. 65-75]. Briefly, the convergence of a sequence $\{u_m\}$ in $L_2^{1,\ell oc}(\Delta, \overline{\Omega})$ is to be proved where $F_m = -(\Delta + z_m)u_m$ has support in Ω_r (defined here by (5.2)). Step 1 is to prove the convergence of $\{u_m\}$ in $L_2^1(\Delta, \Omega_{r''})$ where $r < r'' < r'$. This can be done exactly as in [23]. Step 2 is to prove the convergence of $\{u_m\}$ in $L_2^1(\Delta, \Omega_R - \Omega_{r''})$ for any $R > r''$. The proof in [23, p. 69] makes use of the Green's function for $-\Delta$ in E^3 which is an elementary function. An analogous proof for the waveguide domains considered here would require the Green's functions for $-\Delta$ in each S_i, S'_j and S''_k and these are not known explicitly. As an alternative to this method it will be shown here that there exists a constant $C = C(R)$ such that

for all u_m

$$(5.6) \qquad \|u_m\|_{L_2(\Omega_R - \Omega_{r''})} \leq C \|u_m\|_{L_2(\Omega_{r''})}$$

Convergence of $\{u_m\}$ in $L_2(\Omega_{r''})$ therefore implies its convergence in $L_2^{loc}(\overline{\Omega})$. The remainder of the proof follows that of [23].

To prove (5.6) recall that $u_m = R_\Omega(z_m)F_m$ where $\text{Im } z_m \neq 0$, $\text{supp } F_m \subset \Omega_r$ and r is fixed. Applying Propositions 3.4 and 3.5 to $u_m \in L_2(\Omega)$ gives (cf. (3.46))

$$(5.7) \qquad (u_m)_{i\ell}(y) = c_{i\ell}(z_m) \, e^{i(z_m - \omega_{i\ell}^2)^{1/2} y}$$

$$(5.8) \qquad (u_m)'_{j\ell}(\rho) = c'_{j\ell}(z_m) \, H_\nu^{(1)}((z_m - \omega'^2_{jn(\ell)})^{1/2} \rho), \quad \nu = \nu'_{jm(\ell)}$$

$$(5.9) \qquad (u_m)''_{k\ell}(r) = c''_{k\ell}(z_m) \, r^{-1/2} \, H_\nu^{(1)}(z^{1/2} r), \quad \nu = (\nu_{k\ell}^2 + \tfrac{1}{4})^{1/2}$$

Inequality (5.6) follows from (5.7)-(5.9) and the following

Proposition 5.1. Let $[a,b] \subset (0,\infty)$ and $\sigma_0 > 0$. Then there exist constants $C = C(a,b,\sigma_0)$ and $R = R(a,b,\sigma_0)$ such that

$$(5.10) \qquad r'^{1/2} \left| H_\nu^{(1)}(z^{1/2} r') \right| \leq C \, r^{1/2} \left| H_\nu^{(1)}(z^{1/2} r) \right|$$

for all r and r' such that $R \leq r \leq r'$, all $\nu \geq 1/2$ and all $z \in C$ such that $a \leq \text{re } z \leq b$ and $0 \leq \text{Im } z \leq \sigma_0$.

This estimate follows from F. Olver's uniform estimates for Hankel functions [15, 16].

Proof of Theorem 4. The existence of the limits $v^\pm(x,\omega)$ in $D(\Delta_\Omega^{loc})$ follows from Theorem 3. Hence, only the radiation condition remains to be verified. This follows from (5.7)-(5.9) because $v^\pm(\cdot,\omega) = \lim\limits_{\sigma \to 0+} R_\Omega(\omega^2 \pm i\sigma)g$.

REFERENCES

1. Agmon, S., Lectures on Elliptic Boundary Value Problems, New York:
    Van Nostrand 1965.

2. Bouwkamp, C. J., A note on singularities occuring at sharp edges in
    electromagnetic diffraction theory, Physica 12, 467-474 (1946).

3. Deny, J. and J. L. Lions, Les espaces du type de Beppo Levi, Ann. Inst.
    Fourier (Grenoble) 5, 305-370 (1953-54).

4. Eidus, D. M., The principle of limiting absorption, Mat. Sb. 57, 13-44
    (1962) = A.M.S. Transl. (2) 47, 157-191 (1965).

5. Goldstein, C. I., Eigenfunction expansions associated with the Laplacian
    for certain domains with infinite boundaries. I and II, Trans. Amer.
    Math. Soc. 135, 1-50 (1969); III, 143, 283-301 (1969).

6. Goldstein, C. I., Scattering theory for elliptic differential operators
    in unbounded domains. I, J. Math. Anal. Appl. 45, 723-745 (1974).

7. Goldstein, C. I., Scattering theory in waveguides, in Scattering Theory
    in Mathematical Physics, Ed. J. A. La Vita and J. P. Marchand, Dordrecht:
    D. Reidel 1974.

8. Guillot, J. C., and C. H. Wilcox, Théorie spectrale du laplacien dans
    des ouverts coniques et cylindriques non bornés, C. R. Acad. Sc. Paris
    282, 1171-1174 (1976).

9. Ignatowsky, W., Reflexion elektromagnetischer Wellen an einem Draht,
    Ann. Physik 18, 495-522 (1905).

10. Jones, D. S., The eigenvalues of $V^2u + \lambda u = 0$ when the boundary condi-
    tions are given on semi-infinite domains, Proc. Cambridge Phil. Soc.
    49, 668-684 (1953).

11. Kato, T., Perturbation Theory for Linear Operators, New York: Springer
    1966.

12. Lyford, W. C., Spectral analysis of the Laplacian in domains with
    cylinders, Math. Ann. 218, 229-251 (1975).

13. Magnus, W., F. Oberhettinger and R. P. Soni, Formulas and Theorems for
    the Special Functions of Mathematical Physics, New York: Springer 1966.

REFERENCES (continued)

14. Meixner, J., Strenge Theorie der Beugung elektromagnetischer Wellen an der vollkomen leitenden Kreisscheibe, Z. Naturforsch. 3a, 506-518 (1948).

15. Olver, F. W. J., The asymptotic expansion of Bessel functions of large order, Phil. Trans. Roy. Soc. London, Ser. A 247, 328-368 (1954).

16. Olver, F. W. J., Asymptotics and Special Functions, New York: Academic Press 1974.

17. Rellich, F., Ein Satz über mittlere Konvergenz, Gött. Nachr. (math. phys.) 30-35 (1930).

18. Rellich, F., Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbröhren, in Studies and Essays Presented to R. Courant, New York: Interscience 1948.

19. Skudrzyk, E., The Foundations of Acoustics, New York: Springer 1971.

20. Svešnikov, A. G., The principle of radiation, Dokl. Akad. Nauk SSSR 73, 917-920 (1950).

21. Svešnikov, A. G., The principle of limiting absorption for a waveguide, Dokl. Akad. Nauk SSSR 80, 345-347 (1951).

22. Wilcox, C. H., Initial-boundary value problems for linear hyperbolic partial differential equations of the second order, Arch. Rational Mech. Anal. 10, 361-400 (1962).

23. Wilcox, C. H., Scattering Theory for the d'Alembert Equation in Exterior Domains, Lecture Notes in Mathematics, Vol. 442, New York: Springer 1975.

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Department of Mathematics University of Utah Salt Lake City, Utah 84112	Unclassified
	2b. GROUP
	Not applicable

3. REPORT TITLE

Steady-State Wave Propagation in Simple and Compound ACOUSTIC WAVEGUIDES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Summary Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Wilcox, Calvin H. and J. C. Guillot

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
April 1977	iii + 28	23
8a. CONTRACT OR GRANT NO. N00014-76-C-0276	9a. ORIGINATOR'S REPORT NUMBER(S)	
b. PROJECT NO. Task No. NR-041-370	TR #31	
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*	
d.	none	

10. DISTRIBUTION STATEMENT

Approved for public release, distribution unlimited

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
none	Office of Naval Research Arlington, Virginia 22217

13. ABSTRACT

A limiting absorption principle and corresponding existence theorem is presented for steady-state acoustic wave fields in simple and compound waveguides. The simple guides are semi-infinite cylinders, sectoral horns and conical horns. The compound guides are structures built by coupling one or more simple guides to a bounded cavity. The theory is developed under minimal regularity hypotheses about the waveguide walls. The class of allowable guides includes all the simple, but non-smooth, structures that arise in applications such as guides built from polyhedra and sections of cylinders, cones, spheres and other simple geometrical structures.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Acoustic waveguide						
Cylinder						
Sectoral horn						
Cone						
Steady-state propagation						
Limiting absorption						

DD ,FORM,.1473 (BACK)

(PAGE 2)